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1997 J. Phys. A: Math. Gen. 30 3937

(http://iopscience.iop.org/0305-4470/30/11/021)

J. Phys. A: Math. Gen. **30** (1997) 3937–3945. Printed in the UK PII: S0305-4470(97)80123-0

Generalized point interactions in one-dimensional quantum mechanics

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Received 2 December 1996, in final form 10 February 1997

Abstract. There is a four-parameter family of point interactions in one-dimensional quantum mechanics. They represent all possible self-adjoint extensions of the kinetic energy operator. If time-reversal invariance is imposed, the number of parameters is reduced to three. One of these point interactions is the familiar δ function potential but the other generalized ones do not seem to be widely known. We present a pedestrian approach to this subject and comment on a recent controversy in the literature concerning the so-called δ' interaction. We emphasize that there is little resemblance between the δ' interaction and what its name suggests.

1. Introduction

There is a four-parameter family of point interactions in one-dimensional quantum mechanics [1]. These point interactions represent all possible self-adjoint extensions (SAEs) of the non-relativistic kinetic energy (KE) operator $-(\hbar^2/2m)\,\mathrm{d}^2/\mathrm{d}x^2$. In the following we use units in which $\hbar^2/2m = 1$. A point interaction is such that it is zero everywhere except at the origin $x = 0$. We will review the notion of SAEs in due course. If time-reversal invariance is imposed, as we will see, the number of independent parameters for the point interactions is reduced to three. Unless we mention otherwise, we assume time-reversal invariance. We confine ourselves to one dimension throughout this paper†.

One of these point interactions is the familiar δ function potential

$$V(x) = g\delta(x) \tag{1}$$

where g is a constant parameter. The Schrödinger equation for a stationary state reads as

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{2}$$

where $\psi'' = \mathrm{d}^2\psi/\mathrm{d}x^2$ and other notation is standard. With $V(x)$ of (1), it is understood that the wavefunction $\psi(x)$ is subject to the set of boundary conditions at $x = 0$,

$$\psi(0^+) = \psi(0^-) = \psi(0) \tag{3}$$

$$\psi'(0^+) - \psi'(0^-) = g\psi(0). \tag{4}$$

† We consider the entire x space. One can think of two half-spaces, $x > 0$ and $x < 0$, that are completely disjoint. We do not consider such a situation in this paper.

0305-4470/97/113937+09$19.50

These conditions can be derived by first replacing $\delta(x)$ with a function of a finite width, for example, a function of the square well form, solving the Schrödinger equation, and then letting the width tend to zero. Alternatively one can integrate the left- and right-hand sides of the Schrödinger equation over the interval $(-\epsilon, \epsilon)$ and let $\epsilon \to 0$. In this second derivation, the continuity of ψ, i.e. (3), is an assumption.

At this point let us warn the reader that, unlike finite-range potentials, $V(x)$ of (1) is not a proper operator in Hilbert space. This is in the sense that

$$\int_{-\infty}^{\infty} |V(x)\psi(x)|^2 \, \mathrm{d}x = g^2 \int_{-\infty}^{\infty} |\psi(0)|^2 \, \mathrm{d}x = \infty. \tag{5}$$

That is, $V(x)$ transforms $\psi(x)$ to $V(x)\psi(x)$ which is not normalizable. The $V(x)$ of (1) should be interpreted as a device which leads to the set of boundary conditions (3) and (4). One may think that this is a matter of mathematical pedantry. It can be taken as such for the δ interaction but, as we will see, not for the other generalized point interactions. Even for the δ interaction, this warning is relevant when it is used for the Dirac equation.

The subject of SAEs of the KE operator in one dimension has been discussed extensively in the mathematics-oriented literature [1–5]. We are aware of a few recent papers which dealt with the subject in more physical contexts, namely those by Carreau *et al* [6], by Carreau [7] and by Exner [8]. We are still under the impression, however, that the notion of generalized point interactions in one dimension has not permeated widely among the practitioners of quantum mechanics. A few years ago there was a controversy in the literature concerning the so-called δ' interaction that is one of the point interactions [9–11]. That controversy well illustrated how confusing the subject could be. We do not think the controversy has been clarified satisfactorily.

The purpose of this paper is to present a pedestrian approach to the subject and comment on the above-mentioned controversy on the so-called δ' interaction. This paper has some overlap with, but is complementary to, those of Carreau *et al* [6] and Carreau [7]. We explore the nature of the δ' interaction in more depth. Exner's interest was rather different from ours [8]. He examined models of graph superlattices that involve the δ and δ' interactions. We focus on non-relativistic quantum mechanics, but we will also briefly discuss the relativistic Dirac equation.

In section 2 we review the notion of an SAE of an operator. For the KE operator its SAEs are related to boundary conditions to be satisfied by the wavefunctions at the origin. We then examine the boundary conditions in a general form. In section 3 we examine implications of SAEs of the KE operator in the context of the transmission problem. In this way, for example, it becomes clear why there are three parameters in the SAEs that conform to time-reversal invariance. In section 4 we discuss difficulties in associating the δ' interaction with the derivative of the δ interaction. In section 5 we discuss point interactions for the Dirac equation. A summary is given in section 6.

2. Self-adjoint extensions of the kinetic energy operator

An operator, say A, is defined by specifying its action on every vector in a space or its dense domain that is smaller than the entire space. The adjoint $A^\dagger$ of operator A is defined such that

$$\langle \phi | A\psi \rangle = \langle A^\dagger \phi | \psi \rangle \tag{6}$$

for all ψ and ϕ. Here ψ is in the domain of A and ϕ in the domain of $A^\dagger$. If the two domains coincide and if $A^\dagger = A$, operator A is said to be self-adjoint [12]. In the one-dimensional

case, A is self-adjoint if

$$\int_{-\infty}^{\infty} \phi^* A\psi \, \mathrm{d}x - \int_{-\infty}^{\infty} (A\phi)^* \psi \, \mathrm{d}x = 0 \tag{7}$$

holds for any pair of normalizable wavefunctions $\psi(x)$ and $\phi(x)$ in the same domain.

Let us consider the KE operator

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}. \tag{8}$$

Then (7) can be reduced to

$$-\int_{-\infty}^{\infty} (\phi^*\psi'' - \phi''^*\psi) \, \mathrm{d}x = [\phi^*\psi' - \phi'^*\psi]_{0^-}^{0^+} = 0 \tag{9}$$

where it is understood that $\psi(x)$ and $\phi(x)$ are both twice-differentiable except at $x = 0$. Often it is tacitly assumed that $\psi(x)$ and $\phi(x)$ and their derivatives are all continuous at $x = 0$. In that case (9) obviously holds. Equation (9) itself, however, does not require such continuity at $x = 0$. If we find a set of boundary conditions for ψ and also for ϕ at $x = 0$ such that (9) is satisfied, then we obtain a SAE of the KE operator. Conditions (3) and (4) are such an example. It is easy to confirm that (3) and (4) together with the same in which ψ is replaced with ϕ satisfy (9). The g of (1) can take any real value; thus we obtain a one-parameter family.

We are interested in most general boundary conditions that meet (9). They can be obtained in a variety of forms but let us examine those given by Gesztesy and Kirsh [13] and discussed by Šeba [4]. They are

$$-\psi'(0^+) - \alpha\psi'(0^-) = \beta\psi(0^-) \tag{10}$$

$$-\delta\psi'(0^-) - \gamma\psi(0^-) = \psi(0^+) \tag{11}$$

where α, β, γ and δ are real constants subject to

$$\alpha\gamma - \beta\delta = 1. \tag{12}$$

Note that $\delta\psi'$ is not a variation of ψ'. We have written the boundary conditions for ψ and ψ' but they apply to any other possible wavefunctions. After a little algebra one can confirm that boundary conditions (10)–(12) satisfy (9). Among α, β, γ and δ, three are independent. Thus we have a three-parameter family of SAEs. This is how we define the generalized point interactions. Suppose, furthermore, that the interaction is invariant under space reflection $x \to -x$. This means that the boundary conditions are invariant under $\psi(0^\pm) \to \psi(0^\mp)$ and $\psi'(0^\pm) \to -\psi'(0^\mp)$. This holds if and only if $\alpha = \gamma$.

Let us mention two special cases. Equations (3) and (4) for the δ interaction mean that

$$\alpha = -1 \qquad \beta = -g \qquad \gamma = -1 \qquad \delta = 0. \tag{13}$$

On the other hand the δ' interaction [1–5] is defined by boundary conditions (10)–(12) with

$$\alpha = -1 \qquad \beta = 0 \qquad \gamma = -1 \qquad \delta = -c. \tag{14}$$

This implies that, while $\psi'(x)$ is continuous at $x = 0$, $\psi(x)$ is discontinuous. Note that the δ' interaction so defined is invariant under $x \to -x$ (because $\alpha = \gamma$). This is in contrast to $\delta'(x)$ which is an odd function of x. Here and in the following, by $\delta'(x)$, we mean $\mathrm{d}\delta(x)/\mathrm{d}x$. We think that the naming of the δ' interaction was unfortunate because the interaction has little resemblance to what the name suggests. We will discuss this further as we proceed. In this regard we concur with Exner [8] who said that the name is somewhat misleading. See his comment below equation (2.1).

Carreau *et al* [6] and Carreau [7] used more general boundary conditions which are expressed in terms of four real parameters α, β, ρ and θ. Their α and β are different from those of (10) and (11). If we assume time-reversal invariance, however, one of the four parameters, θ, becomes a trivial one. In their boundary conditions, θ appears in the form of $e^{i\theta}$. Time-reversal invariance requires that ψ^* and ψ'^* satisfy the same boundary conditions as those for ψ and ψ' (with the same coefficients, not their complex conjugates). In other words we should be able to chose ψ to be real for a stationary state. This requires that $e^{i\theta}$ be real, which restricts θ to integral multiples of π. The relevance of time-reversal invariance will become more transparent in the next section.

3. Relation to the transmission problem

In the preceding section we quoted a set of boundary conditions (10) and (11). These are the most general conditions that conform to time-reversal invariance. They contain three real parameters. One may wonder why there are only three parameters and not more. In this regard it is instructive to relate the problem to that of transmission.

When a wave representing a particle is incident on a potential, it is partially transmitted and partially reflected. This can be described in terms of transmission coefficients $T_{\rm L}$ and $T_{\rm R}$ and reflection coefficients $R_{\rm L}$ and $R_{\rm R}$. Suffices L and R refer to the situations in which the wave is incident from the left and right, respectively. The Ts and Rs are in general complex. Hence, they involve altogether eight real parameters. However, unitarity (probability conservation) reduces the number of independent parameters to four. Time-reversal invariance leads to $T_{\rm L} = T_{\rm R}$ and reduces the number of independent parameters to three. Let us assume time-reversal invariance so that $T_{\rm L} = T_{\rm R}$ and suppress suffices L and R of T. If the interactions happen to be symmetric with respect to $x \to -x$, it follows that $R_{\rm L} = R_{\rm R}$ and we are left with only two parameters. Instead of Ts and Rs, we can use the S-matrix which is a 2×2 unitary matrix. Its matrix elements can be expressed in terms of three real parameters, which can be the two eigenphases η_1, η_2 and the mixing parameter ϵ [14–16].

Let us examine the relation between parameters α, β, γ and δ of (10) and (11) and the T and Rs. If the wave is incident from the left, the wavefunction can be written as

$$\psi_{\rm L}(x) = \begin{cases} e^{ikx} + R_{\rm L}\, e^{-ikx} & \text{for } x < 0 \\ T\, e^{ikx} & \text{for } x > 0 \end{cases} \tag{15}$$

where $k = \sqrt{E}$. Imposing (10) and (11) on $\psi_{\rm L}$ and also on $\psi_{\rm R}$ which can similarly be written down, we obtain

$$T = \frac{-2ik}{D} \tag{16}$$

$$R_{\rm L} = \frac{\beta + \delta k^2 + ik(\alpha - \gamma)}{D} \tag{17}$$

$$R_{\rm R} = \frac{\beta + \delta k^2 - ik(\alpha - \gamma)}{D} \tag{18}$$

$$D = -\beta + \delta k^2 + ik(\alpha + \gamma). \tag{19}$$

We have used (12). If we do not impose (12), unitarity does not hold. The equality $R_{\rm L} = R_{\rm R}$ holds if and only if $\alpha = \gamma$. For a finite potential (and also for the δ potential), we know that $T \to 1$ as $k \to \infty$. The T of (16), however, does not necessarily approach 1 as $k \to \infty$.

The eigenphases η_1, η_2 and the mixing parameter ϵ can be expressed in terms of T, $R_{\rm L}$ and $R_{\rm R}$ [16].

There is a bound state with energy $-\kappa^2$ if $D = 0$ for $k = {\rm i}\kappa$ where $\kappa > 0$. Its $\psi(x)$ is of the form of ${\rm e}^{-\kappa x}$ (${\rm e}^{\kappa x}$) for $x > 0$ ($x < 0$), but it is in general discontinuous at $x = 0$. We obtain

$$\frac{\psi(0^+)}{\psi(0^-)} = \alpha + \frac{\beta}{\kappa} = -(\gamma + \delta\kappa). \tag{20}$$

It can be shown that $|\psi(0^+)/\psi(0^-)| = 1$ if and only if $\alpha = \gamma$.

Let us write down the T and Rs for the two special cases of the δ and δ' interactions. For the δ interaction with (13) we find

$$T = \frac{-2{\rm i}k}{g - 2{\rm i}k} \tag{21}$$

$$R_{\rm L} = R_{\rm R} = \frac{-g}{g - 2{\rm i}k}. \tag{22}$$

If $g < 0$ there is a bound state with energy $-\kappa^2$ where $\kappa = -g/2$. Its wavefunction is $\psi(x) = \sqrt{\kappa}\,{\rm e}^{-\kappa|x|}$.

For the δ' interaction with (14) we obtain

$$T = \frac{2{\rm i}}{ck + 2{\rm i}} \tag{23}$$

$$R_{\rm L} = R_{\rm R} = \frac{ck}{ck + 2{\rm i}}. \tag{24}$$

Note that $T \to 0$ and $R \to 1$ as $k \to \infty$. If $c < 0$ there is a bound state with energy $-\kappa^2$ where $\kappa = -2/c$. Its wavefunction is $\psi(x) = \sqrt{\kappa}(x/|x|)\,{\rm e}^{-\kappa|x|}$. This is the only bound state, yet it is of odd parity.

Before ending this section, let us again comment on the boundary conditions used by Carreau *et al* [6] and Carreau [7]. If their ${\rm e}^{{\rm i}\theta}$ is not real, then $T_{\rm L} \neq T_{\rm R}$ which implies that time-reversal invariance does not hold. Unitarity, however, holds even when ${\rm e}^{{\rm i}\theta}$ is not real. We find this feature very interesting in the following sense. Suppose $V(x)$ of (2) is an ordinary finite potential. Time-reversal invariance requires that $V(x)$ is real. Then unitarity holds. On the other hand, unitarity requires that $V(x)$ is real, then time-reversal invariance ensues. Thus time-reversal invariance and unitarity are inseparable for a finite potential. For generalized point interactions, however, we can have a situation such that time-reversal invariance is violated, yet unitarity is valid.

4. The δ' interaction

Let us consider a δ function dipole and define a function $f_\nu(x)$ by

$$f_\nu(x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon^\nu}[\delta(x + \epsilon) - \delta(x - \epsilon)] \tag{25}$$

where $\nu > 0$ is a parameter. The $f_1(x)$ can be interpreted as $\delta'(x)$. Šeba [5] showed very interesting results for the following interaction,

$$V(x) = \lambda f_\nu(x) \tag{26}$$

where λ is a constant.

Let us complement Šeba's highly mathematical analysis with an elementary one for the same interaction (26). Before taking the limit $\epsilon \to 0$, we obtain the following T and Rs,

$$T = \frac{k^2}{\Delta} \tag{27}$$

$$R_{\rm L} = \frac{-h\,{\rm e}^{-2{\rm i}\epsilon k}}{h - {\rm i}k}\left[1 - \frac{k^2\,{\rm e}^{4{\rm i}\epsilon k}}{\Delta}\right] \tag{28}$$

$$R_{\rm R} = \frac{-h\,{\rm e}^{-2{\rm i}\epsilon k}}{h + {\rm i}k}\left[1 - \frac{k^2\,{\rm e}^{4{\rm i}\epsilon k}}{\Delta}\right] \tag{29}$$

$$h = \frac{\lambda}{2\epsilon^{\nu}} \qquad \Delta = k^2 + h^2(1 - {\rm e}^{4{\rm i}\epsilon k}). \tag{30}$$

For this derivation, see example 3 of [16]†. When $\epsilon k \ll 1$ the T and Rs become

$$T = \frac{{\rm i}k}{4\epsilon h^2 + {\rm i}k} \tag{31}$$

$$R_{\rm L} \approx R_{\rm R} = \frac{-4\epsilon h^2}{4\epsilon h^2 + {\rm i}k}. \tag{32}$$

Note that $4\epsilon h^2 = \lambda^2\epsilon^{1-2\nu}$.

If we let $\epsilon \to 0$, the following three situations ensue.

(i) If $\nu > 1/2$, then $\epsilon h^2 \to \infty$ and hence $T \to 0$, $R_{\rm L} \to -1$ and $R_{\rm R} \to -1$. The two subspaces of $x > 0$ and $x < 0$ become effectively disjoint.

(ii) If $\nu = 1/2$, the T and Rs become identical to those due to interaction (1) with $g = -\lambda^2/2$.

(iii) If $\nu < 1/2$, then $\epsilon h^2 \to 0$, $T \to 1$ and the Rs vanish. That is, the interaction disappears.

These three situations are exactly the same as those found by Šeba [5]. As we stated before, $f_\nu(x)$ becomes $\delta'(x)$ when $\nu = 1$. This leads to situation (i) that is not very interesting. It is clear that the δ' interaction does not follow from (25) with $\nu = 1$ or any other value of ν.

In the above it was understood that λ and ν are kept fixed. One can take a flexible attitude and scale λ, for example, in such a way that ϵh^2 is kept constant. This is in the same spirit as the one taken in defining the δ function interactions in three and two dimensions [17, 18]. This, however, does not give anything new; when ϵh^2 is a certain constant, the T and Rs of (31) and (32) are nothing but those of the δ potential (1) with $g = 8\epsilon h^2$. So we end up with situation (ii).

A few years ago Zhao [9] examined the potential

$$V(x) = c\delta'(x) \tag{33}$$

and arrived at the boundary conditions

$$\psi(0^+) = \psi(0^-) = 0 \tag{34}$$

$$\psi'(0^+) - \psi'(0^-) = -\frac{c}{2}[\psi'(0^+) + \psi'(0^-)]. \tag{35}$$

He then criticized the boundary conditions for the δ' interaction of Gesztesy and Holden [3], i.e. (10) and (11) together with (14). Albeverio *et al* pointed out that Zhao's conditions are flawed [9]. Indeed Zhao's conditions do not conform to (10) and (11). If we use

† In each of equations (6.10) and (6.11) of [16], the β^2 in the numerator of the second term of the right-hand side should read as β.

his conditions for the transmission problem we obtain nonsensical results. Albeverio *et al* stressed that one should not take the δ' interaction too literally as Zhao did.

Griffiths [11] also criticized Zhao's conditions and derived different boundary conditions for $V(x)$ of (33). His conditions are

$$\psi(0^+) - \psi(0^-) = \frac{c}{2}[\psi(0^+) + \psi(0^-)] \tag{36}$$

$$\psi'(0^+) - \psi'(0^-) = -\frac{c}{2}[\psi'(0^+) + \psi'(0^-)]. \tag{37}$$

These conditions, which are different from those of the so-called δ' interaction, do conform to (10) and (11) with

$$\alpha = \frac{c-2}{c+2} \qquad \beta = 0 \qquad \gamma = \frac{1}{\alpha} \qquad \delta = 0 \tag{38}$$

therefore, they are acceptable conditions of SAE. They lead to the following T and Rs

$$T = \frac{-2}{\alpha + \gamma} \tag{39}$$

$$R_{\rm L} = -R_{\rm R} = \frac{\alpha - \gamma}{\alpha + \gamma} \tag{40}$$

which are all independent of k.

A question remains, however. Are Griffiths' conditions really correct for $V(x)$ of (33)? If they are, we should be able to obtain the T and Rs given above by using interaction (26) with $\nu = 1$. This is not the case, however, as Šeba showed and we have confirmed. The source of this discrepancy can be traced in Griffiths' derivation of his conditions. In deriving (37) he used

$$\int_{-\epsilon}^{\epsilon} \delta'(x)\psi(x)\,{\rm d}x = -\int_{-\epsilon}^{\epsilon} \delta(x)\psi'(x)\,{\rm d}x = -\frac{1}{2}[\psi'(0^+) + \psi'(0^-)]. \tag{41}$$

We can write $\delta'(x)$ as

$$\delta'(x) = \lim_{\alpha\to 0} \frac{1}{2\alpha}[\delta(x+\alpha) - \delta(x-\alpha)]. \tag{42}$$

It is understood that $0 < \alpha < \epsilon$. For finite α we obtain

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} \delta'(x)\psi(x)\,{\rm d}x &= -\frac{1}{2\alpha}[\psi(\alpha) - \psi(-\alpha)] \\ &= -\frac{1}{2\alpha}[\psi(\alpha) - \psi(0^+) + \psi(0^-) - \psi(-\alpha)] - \frac{1}{2\alpha}[\psi(0^+) - \psi(0^-)].\end{aligned} \tag{43}$$

By letting $\alpha \to 0$, we obtain

$$\int_{-\epsilon}^{\epsilon} \delta'(x)\psi(x)\,{\rm d}x \to -\frac{1}{2}[\psi'(0^+) + \psi'(0^-)] - \frac{1}{2\alpha}[\psi(0^+) - \psi(0^-)]. \tag{44}$$

Hence, (41) is valid only if $\psi(x)$ is continuous at $x = 0$. As Griffiths himself emphasized, however, $\psi(x)$ is not continuous in the situation under consideration.

Instead of (42) one can also use the derivative of a Gaussian function, and arrive at the same conclusion. Equation (44) does not depend on the details of how $\delta'(x)$ is defined. Griffiths' derivation of (36) is also marred by a flaw of the same nature. Griffiths derived conditions for a potential of the form of the nth derivative of the δ function. Let us emphasize that there are only three parameters for the possible point interactions within time-reversal invariance and no room to accommodate potentials of such higher derivatives.

5. Point interactions for the Dirac equation

The δ function interaction (1) used for the non-relativistic Schrödinger equation has no ambiguity. It uniquely implies boundary conditions (3) and (4). When it is used for the Dirac equation, however, $V(x)$ of (1) is not well defined unless the boundary conditions for the wavefunction at $x = 0$ are explicitly specified. In section 1 we mentioned two prescriptions for deriving boundary conditions (3) and (4). If one tries the same prescriptions for the Dirac equation with $V(x)$ of (1) they lead to different sets of boundary conditions. They differ with respect to the effective strength of the interaction [19, 20]. This is a manifestation of the fact that $V(x)$ of (1) is not a proper operator in Hilbert space as we mentioned in section 1.

In view of this complication regarding the δ function potential, one may wonder whether general point interactions for the Dirac equation are very complicated; actually they are not. Again there is a four-parameter family of SAEs of the relativistic KE operator. When time reversal is imposed, the number of parameters is reduced to three. The S-matrix description of the transmission problem also applies to the relativistic case. In one dimension the Dirac wavefunction has two components, ψ_1 and ψ_2. The lower component ψ_2 is essentially the derivative of the upper component ψ_1. The boundary conditions for the Schrödinger equation can be transcribed into those for the Dirac equation by appropriately replacing ψ and ψ' with ψ_1 and ψ_2, respectively.

6. Summary and discussions

We have reviewed the four-parameter family of point interactions in one dimension. They represent all possible SAEs of the KE operator. We have pointed out that, if time-reversal invariance is imposed, the number of parameters is reduced to three. We examined implications of the SAEs in the context of the transmission problem. The existence of the three-parameter family of SAEs that conform to time-reversal invariance is related to the fact that the transmission and reflection coefficients or the S-matrix elements of the transmission problem can be expressed in terms of three real parameters. As far as the SAE of the KE energy is concerned, one-dimensional space is richer than higher dimensions. In each of two- and three-dimensional spaces there is only one parameter family. No SAE is possible in four or more dimensions [1].

The point interactions other than the familiar δ function potential are difficult to visualize. As we said in section 1, the δ function potential $V(x)$ of (1) is not a proper operator in Hilbert space. It is very unlikely that other point interactions can be expressed in terms of acceptable operators. The so-called δ' interaction does not mix the even and odd parity states. This already means that the δ' interaction is different from what one would think of in terms of $\delta'(x) = \mathrm{d}\delta(x)/\mathrm{d}x$ that is an odd function of x. We have examined the $\delta'(x)$ interaction as a limit of δ function dipole. We have presented an elementary derivation of Šeba's results; the $\delta'(x)$ interaction turns out to be equivalent to the δ function interaction or something trivial. We have commented on a recent controversy regarding the δ' interaction. The δ' interaction is the one defined by the boundary conditions (10) and (11) together with (14). It should not be confused with $\delta'(x)$.

We have briefly discussed point interactions for the Dirac equation. Again there is a four-parameter family of point interactions for the Dirac equation. When time-reversal invariance is imposed the number of parameters is reduced to three.

Acknowledgments

We would like to thank Dr M G Calkin, Dr D J Griffiths, Dr D Kiang and Dr Lauro Tomio for helpful discussions and correspondence. One of the authors (YN) is grateful to the Instituto de Física, Universidade de São Paulo and Instituto de Física Teórica, Universidade Estadual Paulista for warm hospitality extended to him during his visit in the Brazilian spring of 1996. This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

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